E9 203 Compressed Sensing & Sparse SP

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Last time:

- Low complexity version of SBL via coordinate descent
- Iteratively reweighted ℓ_1 and ℓ_2 methods for SBL.

Today: Ch. 4 : Basis Pursuit

Recall: $(P_0) : \min_{z \in \mathbb{C}^N} \|z\|_0$ s.t. $y = Az$

NP hard in general

Basis Pursuit: Instead of (P_0), we solve

$$(P_1) : \min_{z \in \mathbb{C}^N} \|z\|_1 \quad \text{s.t. } y = Az$$

Convex opt. pb. ; linear program / QCBP.

Q. When will it "work"?

1. Exact vs. approximate recovery
2. Sparse vs. compressible vectors : Stability
3. Noisy meas. : Robustness
4. $\underbrace{\text{Guarantees } \forall \text{ sparse } x}_{\text{"Uniform" recovery guarantee}}$ vs. $\underbrace{\text{guarantees for a given sparse } x.}_{\text{Nonuniform guarantee}}$ ("Uniform" recovery guarantee $\overset{?}{\Longleftrightarrow}$ Nonuniform guarantee)

Null Space Property (NSP)

[Notation: $v \in \mathbb{C}^N$, $S \subseteq [N]$, $\left.\begin{array}{l} v_S \in \mathbb{C}^{|S|} \\ \text{or } v_S \in \mathbb{C}^N \end{array}\right\}$ Clear from the context.]

Defn. (NSP): $A \in \mathbb{C}^{m \times N}$ satisfies the NSP relative to a set $S \subset [N]$ if

$$\|v_S\|_1 < \|v_{\bar{S}}\|_1 \quad \forall\, v \in \mathcal{N}(A) \setminus \{\underline{0}\}$$

A satisfies the NSP of order s if it satisfies the NSP relative to any $S \subset [N]$ with $|S| \le s$.

Remark: Two reformulations of the NSP:

(i) Add $\|v_S\|_1$ to both sides of the above:

$$2\|v_S\|_1 < \|v\|_1 \quad \forall\, v \in \mathcal{N}(A) \setminus \{\underline{0}\}$$

(ii) Choose S = idx. set of s largest (in magnitude) entries of v, and add $\|v_{\bar{S}}\|_1$ to both sides:

$$\|v\|_1 < 2\,\sigma_s(v)_1 \quad \forall\, v \in \mathcal{N}(A) \setminus \{\underline{0}\}$$

(Recall: $\sigma_s(x)_p \triangleq \inf_{\|z\|_0 \le s} \|x - z\|_p$)

Thm. $A \in \mathbb{C}^{m \times N}$. Every $x \in \mathbb{C}^N$ supported on $S \subset [N]$ is the unique soln. of (P_1) with $y = Ax$ iff A satisfies the NSP relative to S. (NSP(S)).

Proof: $\boxed{\Rightarrow}$

Suppose every $x \in \mathbb{C}^N$, $\text{supp}(x) = S$ is the unique soln to (P_1). That is, $\left. x = \arg\min_{z \in \mathbb{C}^N} \|z\|_1 \text{ s.t. } Az = y = Ax. \right\}$

$\Rightarrow$ For any $v \in \mathcal{N}(A) \setminus \{\underline{0}\}$, v_S is the unique soln. to (P_1) s.t. $Az = Av_S$.

$$A(v_S + v_{\bar{S}}) = Av = 0 \Rightarrow Av_S = -Av_{\bar{S}}$$

Also, $v_S \ne v_{\bar{S}}$ since $v \ne 0$.

So $-v_{\bar{S}}$ is also a "candidate" soln. to (P_1).

By the uniqueness assump., $\|v_S\|_1 < \|v_{\bar{S}}\|_1$

Hence, NSP(S) holds.

$\boxed{\Leftarrow}$ Let $x \in \mathbb{C}^N$ be supported on S.

Let $z \in \mathbb{C}^N$, $z \ne x$ s.t. $Az = Ax$.

(Another candidate soln.)

$v = x - z \in \mathcal{N}(A) \setminus \{\underline{0}\}$.

$v_S = x - z_S$, $v_{\bar{S}} = -z_{\bar{S}}$.

$$\begin{aligned}
\|x\|_1 &\le \|x - z_S\|_1 + \|z_S\|_1 && \triangle\text{le ineq.} \\
&= \|v_S\|_1 + \|z_S\|_1 \\
&< \|v_{\bar{S}}\|_1 + \|z_S\|_1 && [\text{NSP}] \\
&= \|-z_{\bar{S}}\|_1 + \|z_S\|_1 = \|z\|_1
\end{aligned}$$

Hence, z cannot solve (P_1). $\boxtimes$

Thm. $A \in \mathbb{C}^{m \times N}$. Every s-sparse $x \in \mathbb{C}^N$ is the unique soln. of (P_1) with $y = Ax$ iff A satisfies the NSP of order s (NSP(s)).

Remark: The thm. shows that $\forall\, y = Ax$ with s-sparse x, (P_1) solves (P_0) also, when NSP(s) holds.

(Suppose z is the global minimizer of (P_0) with $y = Ax$, where x is the unique sol. of (P_1).

Then, $\|z\|_0 \le \|x\|_0 \Rightarrow z$ is s-sparse

But since A satisfies NSP(s) $\Rightarrow$ Every s-sparse vec is the unique min. of $(P_1) \Rightarrow x = z$.)

Remark: If A satisfies NSP(s), then $\tilde{A}$ satisfies NSP(s) too, where

(i) $\tilde{A} = GA$, with $G \in \mathbb{C}^{m \times m}$ invertible

(ii) $\tilde{A} = \begin{bmatrix} A \\ B \end{bmatrix}$, with $B \in \mathbb{C}^{m' \times N}$ arbitrary.

Stability: What if x is not exactly s-sparse?

(Desirable: recover x w/ err. controlled by $\sigma_s(x)_1$.)

Defn. (SNSP): $A \in \mathbb{C}^{m \times N}$ satisfies the stable NSP (SNSP(ρ)) with const $0 < \rho < 1$ relative to the set $S \subset [N]$ if

$$\|v_S\| \leq \rho \|v_{\bar{S}}\|_1 \quad \forall\, v \in \mathcal{N}(A).$$

It satisfies the SNSP of order s if it satisfies the SNSP relative to the set S for any set $S \subset [N]$ with $|S| \leq s$.

Thm. $A \in \mathbb{C}^{m \times N}$ satisfies the SNSP with const. $0 < \rho < 1$ relative to set S iff

$$(*) \quad \|z - x\|_1 \leq \frac{(1+\rho)}{(1-\rho)} \left[\|z\|_1 - \|x\|_1 + 2\|x_{\bar{S}}\|_1 \right]$$

$\forall\, x, z \in \mathbb{C}^N$ with $Az = Ax$.

Note: Let S = idx set corresp. to s largest (in magnitude) coeffs. in x.

$$\Rightarrow \|x_{\bar{S}}\|_1 = \sigma_s(x)_1$$

Let $x^\#$ be the min. of (P_1) with constraint $Az = Ax$.

$$\|x^\#\|_1 \leq \|x\|_1, \quad Ax^\# = Ax$$

Thus if A satisfies NSP w/ const. $\rho \in (0,1)$ relative to S and if the above thm holds, then $x^\#$ is "candidate" z, implying

$$\|x^\# - x\|_1 \leq \frac{(1+\rho)}{(1-\rho)} \Big(\underbrace{\|x^\#\|_1 - \|x\|_1}_{-ve} + 2 \underbrace{\|x_{\bar{S}}\|_1}_{\sigma_s(x)_1} \Big)$$

$$\boxed{\|x^\# - x\|_1 \leq \frac{2(1+\rho)}{(1-\rho)} \sigma_s(x)_1}$$

$\Rightarrow$ If SNSP is sat., ℓ_1 recovery is "stable" i.e., the error can be controlled by $\sigma_s(x)_1$ ($\&\, \rho$).